MATH4210: Financial Mathematics Tutorial 10

Jiazhi Kang

The Chinese University of Hong Kong

jzkang@math.cuhk.edu.hk

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No Arbitrage Principle

A self-financing portfolio (what is the definition?) $(\Pi_t)_{0 \le t \le T}$, such that

$$\Pi_0 = 0, \ \mathbb{P}[\Pi_t \ge 0] = 1, \text{ and } \mathbb{P}[\Pi_t > 0] > 0$$

for any $t \in [0, T]$, then we say that Π is an arbitrage.
In the context of pricing theories, we always assume the no arbitrage principle: no such self-financing portfolio exists in the market. What happens if such arbitrage opportunities exist (take an example for stock price)?

S_t is the stock. $C(t)$ is the option price associated with S_t

bank + S_t.

Π_t can replicated $C(T)$ s.t. $\forall t \in [t, T)$, $\Pi_t \ge C(t)$

$\Pi'(t)$: buy $C(t)$ and short Π_t

Options

Proposition (Law of One Price)

cash flow

If two portfolios have the same (profit) at maturity time T, then for all prior times $t < T$, the price of the portfolio's must be equal.

Question

Show that the European put options with strike price K and maturity at time T satisfies $\boxed{P_E(t, K) > Ke^{-r(T-t)} - S(t) \text{ for all } t < T}$, where $S(t)$ is the stock price, r is the continuous compounded interest rate.

① Suppose that : $\exists t_0 < T$ s.t. $P_E(t_0) \le Ke^{-r(T-t_0)} - S(t_0)$.

② Construct Π: $\Pi(t_0) = \underbrace{P_E(t_0)}_{\text{buy low}} - \underbrace{\left(Ke^{-r(T-t_0)} - S(t_0)\right)}_{\text{sell high}}$.

long 1 unit $P_E(t_0)$, long 1 $S(t_0)$. short 1 $\boxed{Ke^{-r(T-t_0)}}$

$\Pi(t_0) \le 0.$

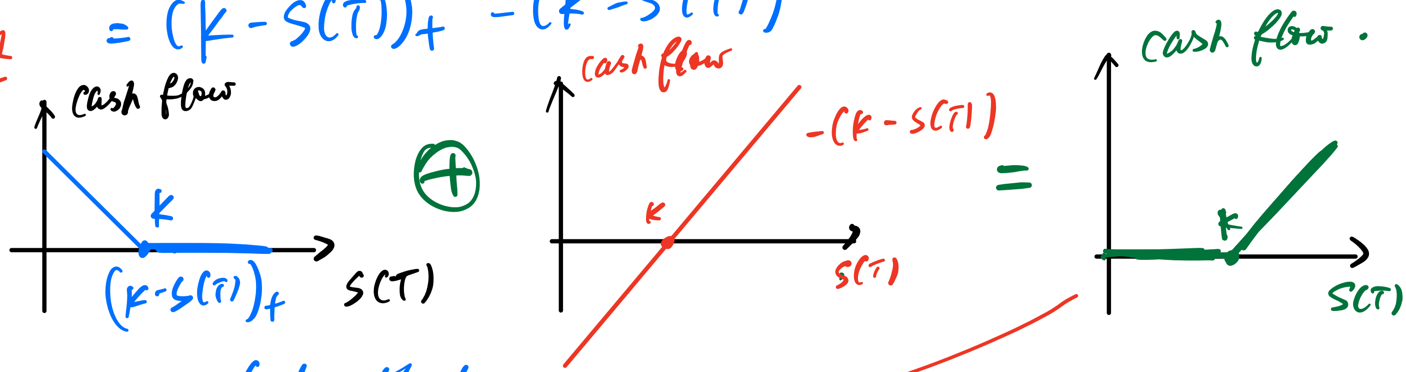
③. Show Π is an arbitrage.

If we arrive T, then

$$\Pi(T) = P_E(T) - (K - S(T))$$

Method 1

$$= (K - S(T))_+ - (K - S(T))$$



We can conclude that

$$\begin{cases} \mathbb{P}[\Pi(T) \ge 0] = 1 \\ \mathbb{P}[\Pi(T) > 0] = \mathbb{P}[S(T) > K] > 0 \end{cases}$$

$\Rightarrow$ Π can define an arbitrage, the contradiction holds.

Method 2 :

$$\Pi(T) = (K - S(T))_+ - (K - S(T))$$

$$= \begin{cases} 0 & \text{if } S(T) \le K \\ S(T) - K & \text{if } S(T) > K. \end{cases}$$

$$\ge 0$$

Options

① Suppose $P_E(t_0, T_1) \geq P_E(t_0, T_2)$ for some $t_0 \leq T_1$

② Construct $\Pi(t_0) = P_E(t_0, T_2) - P_E(t_0, T_1)$ long $P_E(t_0, T_2)$, short $P_E(t_0, T_1)$.

③ Show Π is an arbitrage strategy.

use last question's result. $P_E(t, T) > 0$, and $C_E(t, T) > 0$, $\forall t < T$

Question

Two vanilla put options are identical except for the maturity dates $T_1 < T_2$. If the interest rate is zero between T_1 and T_2, then $P_E(t, T_1) < P_E(t, T_2)$ at any time $t \leq T_1$.

At $t = T_1$, then $\Pi(T_1) = P_E(T_1, T_2) - P_E(T_1, T_1)$.

$$= P_E(T_1, T_2) - (K - S(T_1))_+$$

If $S(T_1) > K$: we don't exercise anything, $\Pi(T_1) = P_E(T_1, T_2) > 0$.

If $S(T_1) \leq K$, then $\Pi(T_1) = P_E(T_1, T_2) - (K - S(T_1)) > K - S(T_1) - (K - S(T_1)) = 0$.

Options

$\boxed{\text{put - call parity.}}$

Question

Suppose two put European options are identical except for the strike prices $0 < K_1 < K_2$, show that

$$0 < C_E(t, K_1) - C_E(t, K_2) < (K_2 - K_1)e^{-r(T-t)}$$

$$0 < P_E(t, K_2) - P_E(t, K_1) < (K_2 - K_1)e^{-r(T-t)},$$

at any time t before maturity T.

① Suppose $P_E(t, K_2) - P_E(t, K_1) \leq 0$ for some $t < T$

Suppose $P_E(t, K_2) - P_E(t, K_1) \geq (K_2 - K_1)e^{-r(T-t)}$ for some $t < T$.

$+S(t)$ $\qquad$ $S(t_D)+D \longrightarrow S(T)+D.$

$t \qquad t_D \qquad T$

Question (Put-Call Parity Relation with Dividend)

Prove the following. Assume that the value of the dividends of the stock paid during $[t, T]$ *is a deterministic constant* D *at time* $t_D \in (t, T]$. *Let* $S(t)$ *be the stock price,* r *be the continuous compounding interest rate ,* $C_E(t, K)$ *and* $P_E(t, K)$ *be the prices of European call and put option at time* t *with strike* K *and maturity* T *respectively. We have*

$$\boxed{C_E(t, K) - P_E(t, K)} = \boxed{S(t) - Ke^{-r(T-t)} - De^{-r(t_D-t)}}$$

$\pi_1(t) = C_E(t,K) - P_E(t,K)$. $\longrightarrow$ $\pi_1(T) = C_E(T,K) - P_E(T,K)$

$= S(T) - K.$

$\pi_2(t) = S(t) - Ke^{-r(T-t)} - De^{-r(t_D-t)}.$

$\rightarrow \pi_2(t_D) = S(t_D) - Ke^{-r(T-t_D)} \cancel{-D+D}$

$= S(t_D) - Ke^{-r(T-t_D)}$

$\Rightarrow \pi_2(T) = \pi_1(T)$

Forward

$\boxed{F(t,T)}$

$\underline{S(T)}$ $\underline{-F(t,T)}$

risky bank

t T

Question

Under no arbitrage opportunity assumptions and assume the continuous compounded interest rate is r, if the stock pays no dividend, show that $F(t,T) = S(t)e^{r(T-t)}$ for $t \geq T$.

$\Pi_1(t)$: long $F(t,T)$, long $F(t,T)e^{-r(T-t)}$

$\Pi_2(t)$: long $S(t)$

$$\Pi_1(T) = (S(T) - \cancel{F(t,T)}) + \left(\cancel{F(t,T)\cdot e^{-r(T-t)}}\right)\cdot \cancel{e^{r(T-t)}}$$

$$= S(T) = \Pi_2(T)$$

$\Rightarrow \forall t \in [0,T]$, $\Pi_1(t) = \Pi_2(t) \Rightarrow F(t,T)e^{-r(T-t)} = S(t)$

$$\Rightarrow F(t,T) = S(t)\cdot e^{r(T-t)}$$

Forward

$\Pi_2(0) = S(0)$: long $S(0)$ at time 0.

$\Pi_2(t) = S(t) + \$ d\, S(t) = (1+d)\, S(t)$

$\Pi_2(T) = (1+d)\, S(T)$

Question

Suppose the stock pay a dividend $d \times S(t)$ at time t, where $0 < t < T$ and $0 < d < 1$, show its forward price $F(0, T) = \dfrac{1}{1+d} S(0) e^{rT}$.

$\Pi_1(0)$. long $(1+d) F(0,T)$, long $(1+d) F(0,T) e^{-rT}$.

$\Pi_1(T) = (1+d)\left(S(T) - \cancel{F(0,T)} \right) + \cancel{(1+d) F(0,T)}$.

$= (1+d)\, S(T) = \Pi_2(T)$.

$\Rightarrow \Pi_1(0) = \Pi_2(0) \Rightarrow (1+d) \cdot F(0,T) e^{-rT} = S(0)$.

$\Rightarrow F(0,T) = \dfrac{1}{1+d} \cdot S(0) \cdot e^{rT}$.